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A second order numerical scheme for a singularly perturbed convection diffusion problem with a non-local boundary condition

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Abstract. This paper deals with the numerical solution of a singularly perturbed convection diffusion problem with a non-local boundary condition. The scheme uses the non-standard finite difference scheme to discretize the derivatives. Using some properties of the discrete operator the stability of the scheme is studied and a first order accuracy is established from the convergence analysis. Richardson extrapolation is then applied on the scheme to increase the first order accuracy to a second order. Numerical experiments are conducted to demonstrate the applicability of the scheme before and after extrapolation.

AMS Subject Classification (2020): 35K20, 65M06, 65M12, 65M15, 93C70

Keywords: Singular perturbations, non-standard finite difference methods, Richardson extrapolation, convergence

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1. Introduction

Consider the singularly perturbed problem

$$\mathcal{L}_\varepsilon u :\equiv \varepsilon u'' + a(x)u' = f(x), \;\; x \in \Omega = (0, l) \tag{1}$$

subject to the non-local boundary condition

$$\varepsilon u'(0) = \mu, \qquad u(0) + \gamma u(l_1) = Bu(l) + d, \;\; l_1 \in \Omega, \tag{2}$$

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where ε is the perturbation parameter and satisfies $\varepsilon \in (0, 1]$. The variables μ, B, γ, d, l_1 and l are constants. The source term $f(x)$ and the coefficient function $a(x)$, are smooth and bounded, $a(x)$ satisfies $a(x) \geq \alpha > 0$ where α is the lower bound of $a(x)$. This ensures that, for small values of the perturbation parameter, the solution $u(x)$ has a boundary layer at $x = 0$.

Generally, the numerical solutions for singularly perturbed problems are not straight forward, because of the multi-scale character which appears their solutions. Even for the least complicated problem classical numerical methods tends to be less efficient.

Researchers in this field either fit the mesh to suit the problem under study or introduce a new coefficient function into the classical numerical schemes to address the issue with unsatisfactory result, see for example the articles [15, 16] and the books [17, 18, 22] for the various fitted numerical methods.

Evidently, the numerical solutions for both stationary and time- dependent singularly perturbed convection diffusion problems with either homogeneous or Dirichlet boundary conditions has been studied extensively, [1, 4, 8, 9, 10, 13, 14, 19, 20, 23]. However same can not be said for problems with non-local boundary conditions, until recently, see for instance the articles [2, 3, 5, 11, 12] and the references there-in.

To the best of our knowledge the non-standard finite difference scheme has only been used by [12] to propose a first order scheme, and again the authors considered an integral boundary. Thus a second order numerical scheme is proposed in this sequel. The scheme employs a fitted operator finite difference scheme on appropriate denominator function to solve the convection diffusion problem.

A first order accuracy independent of the perturbation parameter is obtained in the maximum norm. Richardson extrapolation is then applied to enhance the first order accuracy to a second order.

The paper unfolds as follows: In Section 2, some properties of the continuous problem which show the existence and uniqueness of the solution to problem (1)-(2) are presented. The numerical scheme is presented in Section 3. This is followed by the stability and the convergence of the scheme in Sections 4 and 5 respectively.

In Section 6, Richardson extrapolation is applied on the scheme to enhance the accuracy, followed by the numerical simulations in Section 7 with concluding remarks and future directions of this research being presented in the last section.

2. Bounds on the solution and its derivative

The bounds on the solution to problem (1)-(2) and its derivatives will be derived in this section. The discrete versions of these bounds will play vital roles in the analysis of the numerical methods we shall see in later sections.

Lemma 2.1. (Continuous minimum principle). *Let $\psi(x)$ be a smooth function which satisfies*

$$\psi(x) \geq 0,$$

$$x \in \partial\Omega \text{ and}$$

$$\mathcal{L}_{\varepsilon}\psi(x) \leq 0,$$

$$x \in \Omega.$$

Then $\psi(x) \geq 0 \;\; \forall \; x \; \in \bar{\Omega}$.

Proof. Let (x^*) be a point in Ω such that, $\xi(x^*) < 0$ and

$$\psi(x^*) = \min_{(x) \, \in \, \bar{\Omega}} \psi(x).$$

It is clear that $(x^*) \notin \partial\Omega$. If $(x^*) \in \Omega$, then we have

$$\mathcal{L}_\varepsilon \psi(x^*) = \varepsilon \psi''(x^*) + a(x^*)\psi'(x^*),$$

since $\psi''(x^*) \geq 0$ and $\psi'(x^*) = 0$, thus $\mathcal{L}_\varepsilon \psi(x^*) \geq 0$, which is a contradiction.

Therefore $\Psi(x) \geq 0, \ \forall \ (x) \in \bar{\Omega}$. □

Lemma 2.2. *Suppose a and f are continuous and $1 + \gamma - B \neq 0$. The solution $u(x)$ of the continuous problem and its derivatives satisfy*

$$||u||_\infty \leq C, \tag{3}$$

where

$$\begin{aligned} C = {} & c_0^{-1}[|d| + \alpha^{-1}(|B| + |\gamma|)(|A| + ||f||_1)] \\ & + \alpha^{-1}(|A| + ||f||_1), \end{aligned}$$

with $c_0 = |1 + \gamma - B|$ and $||f||_1 = \int_0^l |f(x)|dx$, and

$$|u^{(j)}(x)| \leq C\left(1 + \varepsilon^{-j}\exp\left(\frac{-\alpha x}{\varepsilon}\right)\right), \quad \forall \ x \in \bar{\Omega}, \tag{4}$$

where j satisfies the relation $1 \leq j \leq 6$.

Proof. To prove the inequality (3), we apply integration factor techniques to equation (1) to obtain

$$\begin{aligned} u'(x) = {} & u'(0)\exp\left(-\varepsilon^{-1}\int_0^x a(\kappa)d\kappa\right) \\ & + \varepsilon^{-1}\int_0^x f(s)\exp\left(-\varepsilon^{-1}\int_s^x a(\kappa)d\kappa\right)ds. \end{aligned} \tag{5}$$

Applying the boundary condition at $x = 0$ gives

$$\begin{aligned} u'(x) = {} & \frac{\mu}{\varepsilon}\exp(-\varepsilon^{-1}\int_0^x a(\kappa)d\kappa) + \varepsilon^{-1} \\ & \int_0^x f(s)\exp(-\varepsilon^{-1}\int_s^x a(\kappa)d\kappa)ds. \end{aligned} \tag{6}$$

Further, we integrate (6) from 0 to x to obtain

$$u(x) = u(0) + \frac{\mu}{\varepsilon}\int_0^x \exp\left(-\varepsilon^{-1}\int_0^K a(\kappa)d\kappa\right)dK$$
$$+\,\varepsilon^{-1}\int_0^x ds f(s)\int_s^x \exp\left(-\varepsilon^{-1}\int_s^K a(\kappa)d\kappa\right)dK. \tag{7}$$

Similar to the proof of Lemma 1 in [3] we take the norm on both sides of equation (7) to obtain

$$|u(x)| \le |u(0)| + |\mu|\alpha^{-1} + \alpha^{-1}\int_0^l |f(s)|ds. \tag{8}$$

To derive the bound of $|u(0)|$ we employ the boundary condition at $x = l$ and the ideas in [3] to obtain

$$|u(0)| \le c_0^{-1}[|d| + \alpha^{-1}\left(|B| + |\gamma|\right)\left(|\mu| + ||f||_1\right)], \tag{9}$$

which leads to (3) when combined with (8).

To prove (4) for $j = 1$, similar to [3] we take the norm on both sides of (6) and simplify further to obtain

$$|u'(x)| \le \frac{|\mu|}{\varepsilon}\exp(-\alpha x/\varepsilon) + \alpha^{-1}||f||_\infty. \tag{10}$$

The proof of the higher order derivatives can be obtained in a similar manner after differentiating equation (1). □

In the next section, a fitted operator finite difference scheme is designed to solve the continuous problem (1)-(2).

3. The numerical method

Here the problem (1)-(2) is discretized via a fitted operator finite difference scheme. The domain Ω is subdivided into the discrete domain

$$\bar{\Omega}^n = \{x_i = ih,\ i = 0, 1, 2, ..., n\}.$$

Now using finite differences and the theory of denominator functions we discretize the continuous problem to obtain the discrete problem

$$\begin{aligned} \mathcal{L}_{\varepsilon}^{n} u_i \;:\equiv\; & \varepsilon\left(\frac{u_{i+1}-2u_i+u_{i-1}}{\phi_i^2}\right)+ \\ & a_i\left(\frac{u_{i+1}-u_i}{h}\right)=f_i, \quad i=1,2,3,...,n-1, \end{aligned} \tag{11}$$

along with the discrete boundary conditions

$$\begin{aligned} \varepsilon\frac{u_0-u_{-1}}{h} &= \mu, \\ u_n &= \frac{1}{B}\left[u_0+\gamma u_{l_1}-d\right]. \end{aligned} \tag{12}$$

The denominator functions ϕ_i^2 is given by

$$\phi_i^2 = \frac{h\varepsilon}{a_i}\left(1-\exp\left(-\frac{a_i h}{\varepsilon}\right)\right),$$

and ϕ_i^2 satisfies

$$\phi_i^2 = h^2+\mathcal{O}\left(\frac{h^3}{\varepsilon}\right).$$

In matrix notation, the scheme (11)-(12) comprises of a tri-diagonal matrix A and two vectors U and F given by

$$AU=F,$$

with the dimensions $\mathbf{R}^n \times \mathbf{R}^n$ and $\mathbf{R}^n$ respectively. Their entries are as follows:

$$\begin{aligned} F_0 &= \mu \\ F_i &= f_i \qquad i=1,2,3,...,n-1, \\ F_n &= f_n-\frac{\varepsilon}{\phi_i^2}u_{n+1} \end{aligned}$$

and

$$A_{ij}=\begin{cases} A_{0,0}=\dfrac{\varepsilon}{h} \qquad A_{0,1}=\dfrac{\varepsilon}{h} \\ r_i^-, \qquad i=2,3,...n-1, j=i-1, \\ r_i^c, \qquad i=1,2,...n-1, i=j, \\ r_i^+, \qquad i=1,2,,...n-2, j=i+1, \end{cases}$$

where the r_i^-, r_i^c and r_i^+ are given by

$$r_i^- = \frac{\varepsilon}{\phi_i^2},$$
$$r_i^c = -2\frac{\varepsilon}{\phi_i^2} - \frac{a_i}{h},$$
$$r_i^+ = \frac{\varepsilon}{\phi_i^2} + \frac{a_i}{h},$$

respectively.

Next we prove some properties of the scheme which will play a vital role in the in the analysis of the scheme in subsequent sections.

4. Stability of the Scheme

In this section, two lemmas which indicate the stability of the scheme are presented.

Lemma 4.1. (Discrete minimum principle). *Let Ψ_i be a mesh function which satisfies*

$$\Psi_0 \geq 0,$$
$$\Psi_n \leq 0 \text{ and}$$
$$\mathcal{L}_\varepsilon^n \Psi_i \leq 0, \;\; i = 1, 2, ...n-1$$

then $\Psi_i \leq 0, \;\; \forall i$.

Proof. Let $j \in \bar{\Omega}^n$ such that

$$\Psi_j = \min_{(i)\in\bar{\Omega}^n} \Psi_i \;\text{ and }\; \Psi_j < 0.$$

Then $1 \leq j \leq n-1$ and $\Psi_{j+1} - \Psi_j > 0, \;\Psi_j - \Psi_{j-1} > 0$. Thus

$$\begin{aligned}\mathcal{L}_\varepsilon^n \Psi_j &\equiv \frac{\varepsilon}{\phi_i^2}(\Psi_{j+1} - 2\Psi_j + \Psi_{j-1}) + a_j \frac{\Psi_{j+1} - \Psi_j}{h} \\ &= \frac{\varepsilon}{\phi_i^2}[(\Psi_{j+1} - \Psi_j) - (\Psi_j - \Psi_{j-1})] + a_j \frac{\Psi_{j+1} - \Psi_j}{h} \geq 0,\end{aligned}$$

which is a contradiction, therefore, $\Psi_j \geq 0$, and hence $\Psi_i \geq 0, \;\forall\, i$. □

An immediate consequence of this discrete minimum principle is the stability of the scheme (11)-(12) which is presented in the next lemma.

Lemma 4.2 (Stability estimate). *The solution u_i of the discrete problem (11)-(12) satisfies the estimate*

$$|u_i| \leq \alpha^{-1} \max_{i\in\bar{\Omega}^n} |\mathcal{L}_\varepsilon^n u_i| + \max_{i\in\bar{\Omega}^n} \left(|\mu|, |\mathcal{B}|\right), \quad \mathcal{B} = B^{-1}\left[u_0 + \gamma u_{l_1} - d\right].$$

Proof. Let $\Psi_i^{\pm}$ be a barrier function given by

$$\Psi_i^{\pm} = M \pm u_i,$$

where M is given by

$$M = \alpha^{-1} \max_{i\in\bar{\Omega}^n} |\mathcal{L}_\varepsilon^n u_i| + \max_{i\in\bar{\omega}^n}(|\mu|, |\mathcal{B}|).$$

At the boundaries we have

$$\Psi_0^{\pm} = M \pm u_0 = M \pm \mu \geq 0,$$
$$\Psi_n^{\pm} = M \pm u_n = M \pm \mathcal{B} \geq 0.$$

On the domain Ω^n we obtain

$$\mathcal{L}_\varepsilon^n \Psi_i^{\pm} = \alpha^{-1} \max_{i\in\bar{\Omega}^n} |\mathcal{L}_\varepsilon^n u_i| \pm \mathcal{L}_\varepsilon^n u_i \leq 0.$$

From Lemma 4.1, $\Psi_i \geq 0, \ \forall \ (x_i) \in \bar{\Omega}^n$, as required. □

The error associated with the above disrectization process is estimated in the next section.

5. Error estimate

Let U_i and u_i as the numerical and the exact solutions of the discrete problem (11)-(12). The truncation error of the scheme (4)-(5) is given by

$$\begin{aligned}
\mathcal{L}_\varepsilon^n(U_i - u_i) &= \mathcal{L}_\varepsilon^n U_i - \mathcal{L}_\varepsilon^n u_i \\
&= f_i - \mathcal{L}_\varepsilon^n u_i = \mathcal{L}_\varepsilon u_i - \mathcal{L}_\varepsilon^n u_i \\
&= \varepsilon u_i'' + a_i u_i' - \left[\varepsilon\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\phi_i^2}\right)\right. \\
&\quad \left. + a_i\left(\frac{u_{i+1} - u_i}{h}\right)\right], \ i = 1, 2, \cdots, n-1. \qquad (13)
\end{aligned}$$

Using a truncated Taylor series expansions of the terms u_{i+1} and u_{i-1} reduces equation (6) into

$$\mathcal{L}_\varepsilon^n(U_i - u_i) = \varepsilon u_i'' - \frac{\varepsilon}{\phi_i^2}\left(h^2 u_i^{''} + \frac{h^4}{12}u_i^{iv} + ...\right) - a_i \frac{h}{2}u_i'' + ...$$

Applying series expansion of the denominator function and further simplification yields

$$\mathcal{L}_\varepsilon^n(U_i - u_i) = -\frac{\varepsilon}{2.3!}h^2 u_i^{iv} - \frac{a_i}{2}h u_i'' - \frac{a_i}{4!}u^{iv} + ... - a_i \frac{h}{2}u_i''$$

From $a(x) > \alpha$, Lemma 2.2 and observing that as $\varepsilon \to 0$ all the exponential terms vanishes (see [20] for proof) we obtain

$$|\mathcal{L}_\varepsilon^n(U_i - u_i)| \leq Ch.$$

By Lemma 4.2 we obtain the result

$$|U_i - u_i| \leq Ch, \qquad (14)$$

where C is a constant independent of the perturbation parameter ε.

The error at the boundaries is estimated as follows:

$$\begin{aligned}
U_0 - u_0 &= -\varepsilon u_x(0) - \mu \\
&\quad \varepsilon u(0) - \left(\varepsilon\left(\frac{u_1 - u_0}{h}\right)\right).
\end{aligned}$$

Applying Taylor series expansions yields

$$U_0 - u_0 = \varepsilon u_0' - \varepsilon\left(u_0' + \frac{h}{2}u_0'' + \frac{h^2}{3!}u_0'' + ...\right) \qquad (15)$$

Simplifying further and then applying Lemma 2.2 yields

$$|U_0 - u_0| = -\varepsilon \frac{h}{2} u_0'' - \varepsilon \frac{h^2}{3!} u_0'' + ... \leq Ch.$$

We summarize the above discussion in the following theorem.

Theorem 5.1. *The solution u_i of the scheme 11-12 satisfies*

$$\sup_{0<\varepsilon\leq 1} \max_{0\leq i\leq n} |U_i - u_i| \leq Ch,$$

where C is a constant independent of ε and h.

In the next section Richardson extrapolation is employed to increase the order of the fitted operator finite difference scheme.

6. Richardson extrapolation

Suppose Ω^{2n} satisfies $\Omega^n \subset \Omega^{2n}$ where Ω^n is as defined in Section 3 and Ω^{2n} obtained after the mesh widths h has been halved. We denote the numerical solution obtained with the mesh Ω^{2n} by $\tilde{U}_i$. From Theorem 5.1, we have

$$U_i - u_i \leq Ch + \bar{R}^n, \tag{16}$$

$$\tilde{U}_i - u_i \leq C\left(\frac{h}{2}\right) + \bar{R}^{2n}, \tag{17}$$

where $\bar{R}^\nu$ is the remainder term of the error. Subtracting the inequality (17) from (16) gives the extrapolation formula

$$U_i - u_i - (2\tilde{U}_i - 2u_i) = \bar{R}_i^n - 2\bar{R}_i^{2n} \tag{18}$$

$$\mathcal{U}_i = 2\tilde{U}_i - u_i, \tag{19}$$

notice that $\mathcal{U}_i$ is the numerical solution after extrapolation at the interior mesh point.

To compute the numerical solution $\mathcal{U}_0$ at the boundary $x = 0$ we use the extrapolation formula,

$$\mathcal{U}_0 = 2\tilde{U}_0 - U_0 \tag{20}$$

and to calculate the error we use the formula

$$\begin{aligned}\mathcal{U}_0 - u_0 &= 2\left[\varepsilon u_0' - \varepsilon\left(u_0' + \frac{h}{4}u_0'' + \frac{h^2}{4.3!}u_0'' + ...\right)\right] - (U_0 - u_0)\\ &= 2\left[-\varepsilon\left(\frac{h}{4}u_0' + \frac{h^2}{4.3!}u_0'' + ...\right)\right] - \varepsilon\left(\frac{h}{2}u_0'' + \frac{h^2}{3!}u_0'' + ...\right).\end{aligned}$$

Further simplification yields

$$|\mathcal{U}_0 - u_0| \leq Ch^2. \tag{21}$$

At the interior mesh points, the error after the extrapolation satisfies

$$\mathcal{L}_\varepsilon^n(\mathcal{U}_i - u_i) \quad = \quad 2\mathcal{L}_\varepsilon^n(\tilde{U}_i - u_i) - \mathcal{L}_\varepsilon^n(U_i - u_i)$$

From series expansions and simplifications the error reduces to

$$\begin{aligned}\mathcal{L}_\varepsilon^n(\mathcal{U}_i - u_i) = 2&\left[\frac{\varepsilon}{48}u_i^{iv}h^2 - \frac{a_i}{4}hu_i'' - \frac{a_i}{8.24}h^3u_i^{iv} - a_i\left(\frac{h}{4}u_i'' + ...\right)\right]\\ &- \left(\frac{\varepsilon}{12}u_i^{iv}h^2 - \frac{a_i}{2}hu_i'' - \frac{a_i}{24}h^3u_i^{iv} - a_i\left(\frac{h}{2}u_i'' + ...\right)\right).\end{aligned}$$

Simplifying further and applying Lemma 4.2 gives the main result in the theorem below.

Theorem 6.1. *Let u_i be the exact solution of the continuous problem (1)-(2) and $\mathcal{U}_i$ be the numerical solution obtained via (18). Then the error is of the form*

$$\sup_{0<\varepsilon\leq 1}\max_{0\leq i\leq 2n}|\mathcal{U}_i - u_i| \leq Ch^2.$$

Next, we use a test example to demonstrate the scheme in practice.

7. Numerical Results

In this section, we apply the scheme on a test example to check the applicability of the proposed scheme. The numerical result will comprise of computing the maximum pointwise error and the numerical rate of convergence in each example.

In the example which follows, to compute the maximum pointwise errors E_ε^n and $\mathcal{E}_\varepsilon^n$, before and after extrapolation respectively , we employ the formulae

$$E_\varepsilon^n = \max_{0\le i\le n} |U_i - u_i|, \tag{22}$$

and

$$\mathcal{E}_\varepsilon^n = \max_{0\le i\le 2n} |\mathcal{U}_i - u_i|. \tag{23}$$

Notice that the subtraction is between the numerical solution and the exact solution respectively, in both before extrapolation and after extrapolation formulae.

From the maximum pointwise errors, we obtain the ε-uniform pointwise errors E_n and $\mathcal{E}_n$ before and after extrapolation by

$$E_n = \max_{0<\varepsilon\le 1} E_\varepsilon^n$$

and

$$\mathcal{E}_n = \max_{0<\varepsilon\le 1} E_\varepsilon^n.$$

Further the numerical rate of convergence is calculated from the formulae

$$r_1 = \log_2\left(\frac{E_n}{E_{2n}}\right) \text{ and } r_2 = \log_2\left(\frac{\mathcal{E}_n}{\mathcal{E}_{2n}}\right)$$

and the ε-uniform rate of convergence with

$$R^1 = \max_{0<\varepsilon\le 1} r_1, \;\; \mathcal{R} = \max_{0<\varepsilon\le 1} r_2,$$

respectively.

Example 7.1 [6]. We consider the problem

$$\begin{aligned} \varepsilon u''(x) &+ 2u'(x) = (\varepsilon - 2)exp(-x), \;\; 0 < x < 1, \\ u(0) &= \frac{1}{\varepsilon}, \quad u(0) - \frac{1}{3}u\left(\frac{1}{4}\right) = 1. \end{aligned}$$

The exact solution is given by,

$$u(x) = p_1 + p_2 + e^{-2x/\varepsilon} + e^{-x},$$

where

$$p_1 = -\frac{3}{7}\left[e^{-1} + \frac{1}{3}e^{-1/4} + \left(1 + e^{-2/\varepsilon} + \frac{1}{3}e^{-1/(2\varepsilon)}\right)p_2\right]$$

and

$$p_2 = -\frac{1+\varepsilon}{2}.$$

Table 1: Maximum pointwise error and rate of convergence for Example 7.1 before extrapolation

ε	n=16	32	64	128	256	512	1024
10^{-2}	1.51E-02	6.42E-03	2.11E-03	5.87E-04	1.51E-04	3.81E-05	9.53E-06
	1.2348	1.6026	1.8467	1.9586	1.9893	1.9971	
10^{-4}	1.80E-02	9.41E-03	4.80E-03	2.41E-03	1.20E-03	5.84E-04	2.77E-04
	0.9364	0.9722	0.9932	1.0108	1.0344	1.0780	
10^{-6}	1.80E-02	9.45E-03	4.83E-03	2.44E-03	1.23E-03	6.16E-04	3.08E-04
	0.9341	0.9676	0.9840	0.9922	0.9964	0.9987	
10^{-8}	1.80E-02	9.45E-03	4.83E-03	2.44E-03	1.23E-03	6.16E-04	3.08E-04
	0.9341	0.9676	0.9840	0.9922	0.9964	0.9987	
10^{-10}	1.80E-02	9.45E-03	4.83E-03	2.44E-03	1.23E-03	6.16E-04	3.08E-04
	0.9341	0.9676	0.9840	0.9922	0.9964	0.9987	
10^{-12}	1.80E-02	9.45E-03	4.83E-03	2.44E-03	1.23E-03	6.16E-04	3.08E-04
	0.9341	0.9676	0.9840	0.9922	0.9964	0.9987	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10^{-20}	1.80E-02	9.45E-03	4.83E-03	2.44E-03	1.23E-03	6.16E-04	3.08E-04
	0.9341	0.9676	0.9840	0.9922	0.9964	0.9987	
E_n	1.80E-02	9.45E-03	4.83E-03	2.44E-03	1.23E-03	6.16E-04	3.08E-04
R^1	0.9341	0.9676	0.9840	0.9922	0.9964	0.9987	

Table 2: Maximum pointwise error and rate of convergence for Example 7.1 after extrapolation

ε	n=16	32	64	128	256	512	1024
10^{-6}	9.30E-05	2.45E-05	6.27E-06	1.59E-06	3.99E-07	1.00E-07	2.51E-08
	1.9133	1.9122	1.7923	1.4129	0.7810	0.2857	
10^{-8}	9.30E-05	2.45E-05	6.28E-06	1.59E-06	4.03E-07	1.03E-07	2.82E-08
	1.9264	1.9633	1.9799	1.9826	1.9622	1.8715	
10^{-10}	9.30E-05	2.45E-05	6.27E-06	1.59E-06	3.99E-07	1.00E-07	2.51E-08
	1.9265	1.9638	1.9820	1.9910	1.9952	1.9964	
10^{-12}	9.30E-05	2.45E-05	6.27E-06	1.59E-06	3.99E-07	1.00E-07	2.51E-08
	1.9265	1.9638	1.9820	1.9910	1.9952	1.9964	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	9.30E-05	2.45E-05	6.27E-06	1.59E-06	3.99E-07	1.00E-07	2.51E-08
	1.9265	1.9638	1.9820	1.9910	1.9952	1.9964	
10^{-24}	9.30E-05	2.45E-05	6.27E-06	1.59E-06	3.99E-07	1.00E-07	2.51E-08
	1.9265	1.9638	1.9820	1.9910	1.9952	1.9964	
10^{-28}	9.30E-05	2.45E-05	6.27E-06	1.59E-06	3.99E-07	1.00E-07	2.51E-08
	1.9265	1.9638	1.9820	1.9910	1.9952	1.9964	
$\mathcal{E}_n$	9.30E-05	2.45E-05	6.27E-06	1.59E-06	3.99E-07	1.00E-07	2.51E-08
$\mathcal{R}^1$	1.9265	1.9638	1.9820	1.9910	1.9952	1.9964	

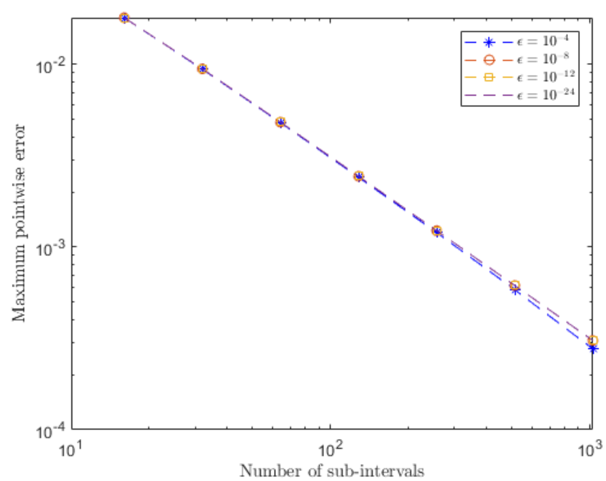

Fig. 1: Loglog plot of the maximum pointwise errors of Example 7.1 before extrapolation

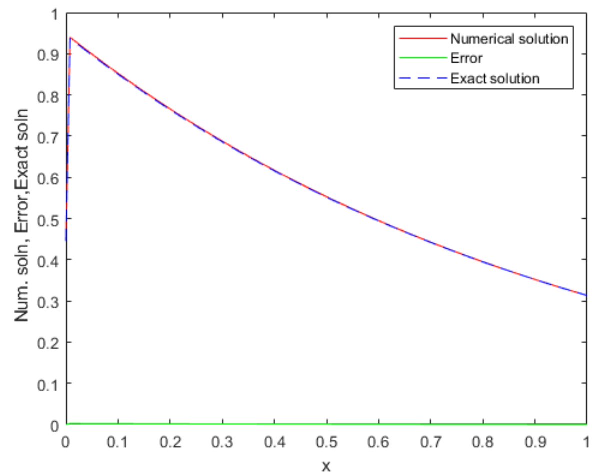

Fig. 2: Plot of the exact, numerical solution and the error of Example 7.1 for $n = 128$ and $\varepsilon = 10^{-24}$

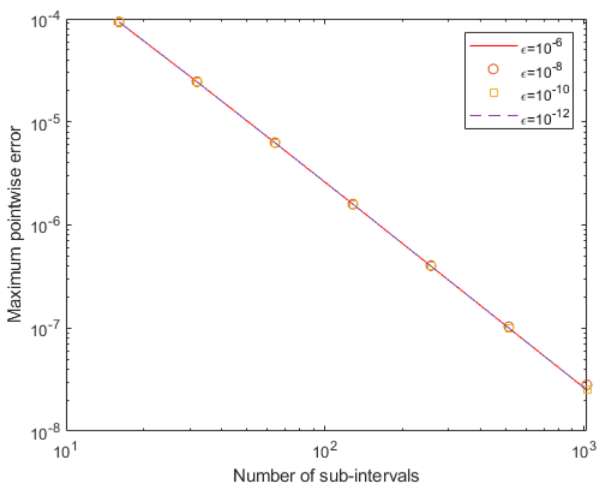

Fig. 3: Loglog plot of the maximum pointwise errors of Example 7.1 after extrapolation

8. Conclusion

A second order numerical scheme which is independent of the perturbation parameter has been proposed in this paper to solve a singularly perturbed convection diffusion problem with non-local boundary conditions. The scheme employed a fitted operator finite difference scheme for the discretization and analysed it for convergence.

The analysis resulted in a first order accuracy and later improved to a second order by the application of Richardson extrapolation. The scheme was then tested on an example to demonstrate it in practice and the results were presented in Tables 1 and 2.

In each table, the maximum pointwise error and the numerical rate of convergence for different values of ε and n were displayed. The rates of convergence in the Tables 1 and 2 were in accordance with the Theorems 5.1 and 6.1, respectively.

Figures 1 and 3 displayed the log-log plot of the maximum pointwise errors before and extrapolation. Clearly, the errors were independent of ε. Also, from Figure 2, one can confirm that there is no significant difference between the numerical solutions and the exact solution since the errors were on the zero mark.

Currently the scheme is being tested on delay problems with non-local boundary conditions.

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